Such growth simply would lead to chaos if there were not adequate advances in all-weather flying capability, air traffic control, noise abatement, air terminal development and ground transport, or more probably, would not take place at all!

As we said at the outset, reservations about future predictions are proper and in order. Yet, simultaneous progress in all these areas hinges on a profound understanding of what the future might hold. This is an understanding that has to be developed before the fact, developed now, not then.

Since this paper was first presented, there has been much discussion of it. In general, little or no resistance to the size of the projected traffic volume is in evidence. There are, in fact, other equally serious projections predicting greater growth than this study. However, it has been rather difficult for people to accept the large numbers of airplanes associated with such a traffic volume. The fleet size will indeed change as different fleet mixes are assured, but not much, except as much larger inputs of SST's and very large new subsonic transports are assumed. It may seem presumptuous to make predictions for as far ahead as 1980 but, in terms of the number of significantly advanced aircraft that can be injected into the fleet, 1980 isn't very far away.

Conclusions

Even with advanced transport vehicles of much greater productivity than that of current transports included in the fleet, the future growth of the air transportation business will require large numbers of aircraft by comparison with the present total fleet. The technology is at hand to make feasible vehicles of much greater productivity. These more-productive vehicles will operate at significantly reduced operating costs.

A total air transportation system on the scale here projected for the 1980's has implications far beyond the vehicles themselves. Airline operating techniques, regulatory considerations, international agreements, airway and ground facilities, and many other vital elements of the total system will require careful integration to permit this expansion.

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Linear Theory of Superbooms Generated by Refraction

M. K. MYERS* AND M. B. FRIEDMAN†
Columbia University, New York, N. Y.

An analysis is made of the flow past a slender body of revolution, moving at supersonic velocity, in an atmosphere in which the speed of sound decreases linearly with altitude. The problem is treated in the context of linearized theory. The primary interest is directed to the pressure distribution occurring in regions where the signal generated by the body is focussed, because of refraction, in the nonuniform medium. Explicit expressions are derived for this pressure signature, and it is found that the rise in pressure behind the Mach surface in the nonuniform atmosphere at the focal distance is of the order of four times that occurring behind the Mach cone in a uniform atmosphere at similar distances from the moving body.

Nomenclature

a = sound speed gradient
$c(z)$ = speed of sound
c_0, c_g = speed of sound at body and at ground
${}_i(z), i = 1,2,3$ = functions expressing envelopes in plane $y = 0$; Eq. (12)
$G_{c1}, G_{c2}, G_{v1}, G_{v2}$ = form functions for pressure signature at cusp level
$H(t)$ = unit step function
K = source strength
L = body length
$m(x)$ = source strength per unit length
$M(z)$ = Mach number: $M(z) = M_0/(1 - \epsilon z)$
M_0 = Mach number at $z = 0$
N = number of roots of Eq. (8)
p = pressure due to single source
P = pressure due to source distribution
$R_0(x)$ = radius of slender body
$S(x)$ = slender body cross-sectional area
t = time
t_i = roots of Eq. (13)
$t_{fi}, i = 1,2$ = roots of Eq. (13) corresponding to envelopes $f_i(z)$
t^* = root of Eq. (13) corresponding to cusp point
t_0 = thickness ratio of slender body
T_i, $i = 1,2,3$ = dimensionless roots of Eq. (13): $T_i = \epsilon c_0 t_i$
V = speed of moving body
x,y,z = coordinates fixed in moving body
α = dimensionless time difference: $\alpha = \epsilon c_0(t_{fi} - \theta)$
β = $\beta^2 = M_0^2 - (1 - \epsilon z)^2$
β_0 = $\beta_0^2 = M_0^2 - 1$
Γ = coefficient in cusp level pressure distributions
$\delta(x_1, \ldots, x_n)$ = Dirac delta function in n dimensions
δ = distance away from Mach envelopes
Δ = dimensionless distance from leading Mach envelope
ϵ = parameter expressing sound speed gradient
$\bar{\epsilon}$ = dimensionless parameter: $\bar{\epsilon} = \epsilon L$
θ = elapsed time after source disturbance
ν = function defining elementary fronts, Eq. (4)
ρ_1, ρ_2 = $\rho_1^2 = \xi^2 + \eta^2 + \zeta^2$, $\rho_2^2 = \rho_1^2 + 4(1 - \epsilon\zeta)/\epsilon^2$
ξ, η, ζ = coordinates fixed in space
σ_i, $i = 1,2,3$ = function defined in Eq. (25)
τ = instant of source disturbance

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* Assistant Professor of Civil Engineering, Institute of Flight Structures, Department of Civil Engineering and Engineering Mechanics. Member AIAA.

† Professor of Civil Engineering, Institute of Flight Structures, Department of Civil Engineering and Engineering Mechanics.

Subscripts

I = subscript denoting pressure due to instantaneous signal
II = subscript denoting pressure due to tail effect

I. Introduction

THE solution for the pressure field, generated by a slender body of revolution moving at a constant supersonic velocity in a homogeneous acoustic medium, is well known. It provides an adequate description of the aerodynamic forces experienced by the body. However, interesting and important far field phenomena are known to occur which cannot be accounted for without the presence of some inhomogeneity in the medium. The objective of this work is to analyze the flow past a slender body of revolution moving at supersonic velocity in an atmosphere in which the velocity of sound in the medium at rest decreases linearly with altitude. The fluid medium is assumed to be nonviscous and nonconducting, and disturbances are assumed sufficiently small that the flow may be treated in the context of linearized (acoustic) theory.

It is known that for short pulses, the perturbation pressure satisfies a wave equation with a varying sound speed provided that the atmosphere at rest is in convective equilibrium.[1] For an ideal gas, this implies a linear decrease with altitude of the square of the sound speed.[2] In the present problem, the small pressure disturbance generated by the body is considered to satisfy a wave equation, with the sound speed as a linear function of the vertical coordinate. With a gradient appropriate to the standard atmosphere (up to 40,000 ft), this is a close approximation to the more exact function required for convective equilibrium over the range of interest, which is between the altitude of the moving body and the flight track.[3] The disturbance field above the body in the direction of decreasing sound speed does not affect that below the body. The presence of a sonic altitude gives rise to a propagation of disturbances upstream from the body in spite of the fact that the body travels at supersonic speed at its altitude. Hence, it is not possible to solve the present problem by assuming the existence of a steady state in body coordinates with undisturbed flow upstream from the Mach envelope. Rather, it is necessary to determine the transient behavior of the system of pressure waves emitted over the entire time history of the motion.

The solution for the pressure field is obtained by representing the body by an axial distribution of point sources that move impulsively from rest at a supersonic velocity. The source solution to the wave equation with a linear sound speed is derived and employed for this purpose. Only motion normal to the sound speed gradient is considered, although the solution may be modified to take account of a vertical component of stream velocity.

Among the consequences of the present model is a phenomenon of major interest: the formation of cusps on the characteristic surfaces (Mach envelopes). In particular, on the flight track directly below the body, the cusp level occurs where the freestream Mach number equals 1. Cusps are formed also by acceleration and turning of a body in a uniform medium; but in the present case, they arise from refraction of the signals generated by the body as they propagate through regions of increasing sound speed. The resulting localized amplification is an important feature of the general problem of sonic boom. Ray theory (geometrical acoustics) has been applied to the boom problem, and the geometrical results, particularly for the linearly varying sound speed, have been used with considerable success to predict the location of the shock front and the location of superbooms.[4–6] However, the locus of cusps is a caustic of the ray system associated with the moving body, and it is well known that the geometric acoustics prediction of the field becomes invalid near a caustic.[1] The theory derived in the present paper, based on a full linear solution, provides simple expressions that describe the behavior of acoustic disturbances at the caustic in terms of pressure levels several times those of the uniform case.

It is not possible, of course, within the context of a linearized theory, to describe exactly phenomena involving shocks. However, the flow is sonic at the cusp, and therefore the strength of any shock propagating into this region must tend to zero.[7] Hence, the results obtained in this analysis should be a valid approximation for the pressure signature near the cusp level. In any case, the linear theory derived here represents the first step toward a corrected theory to account for the decay of the shock in the neighborhood of the cusp.

II. Solution for Stationary Source

Consider the propagation of sound from a stationary point source located at some depth below the surface of a half space, in which the speed of sound increases linearly with depth. Let a rectangular coordinate system (ξ,η,ζ) be introduced, and let the coordinates of the source point be denoted by (ξ_0,η_0,ζ_0). The speed of sound in the half space is described by $c(\zeta) = c_0 - a(\zeta - \zeta_0)$, where c_0 denotes the speed of sound at the depth of the source, and a is the gradient of the sound speed. The surface of the half space corresponds to the level at which $c(\zeta) = 0$. It is convenient to take $\xi_0 = \eta_0 = \zeta = 0$ and to introduce a parameter ϵ such that $c(\zeta)$ may be written

$$c(\zeta) = c_0(1 - \epsilon\zeta) \tag{1}$$

where c_0 is the speed of sound at the source level $\zeta = 0$.

The field generated by the source in the half space may be described by a pressure perturbation $p'(\xi,\eta,\zeta,t)$, which satisfies the equation

$$\nabla^2 p' - \{1/[c_0^2(1 - \epsilon\zeta)^2]\}p_{tt}' = -\delta(\xi,\eta,\zeta)\delta(t) \tag{2}$$

for $-\infty < \xi,\ \eta < \infty,\ -\infty < \zeta \leq 1/\epsilon, 0 \leq t$, where $\delta(t)$ is the Dirac delta. Among the solutions of Eq. (2), the source is characterized by the conditions that p' represents an outgoing wave and is bounded everywhere except in the neighborhood of $\xi = \eta = \zeta = t = 0$.

The desired solution can be obtained from Eq. (2) by the formal application of the Laplace and Hankel transforms in succession. The result, derived in Ref. 8, may be written

$$p' = \frac{(1 - \epsilon z)^{1/2}}{2\pi\epsilon\rho_1\rho_2}\left\{\delta\left(t - \frac{\nu}{\epsilon c_0}\right) - \frac{\nu J_1[(\epsilon c_0/2)(t^2 - \nu^2/\epsilon^2 c_0^2)^{1/2}]}{2(t^2 - \nu^2/\epsilon^2c_0^2)^{1/2}} H\left(t - \frac{\nu}{\epsilon c_0}\right)\right\} \tag{3}$$

where

$$\nu = \cosh^{-1}[1 + \epsilon^2\rho_1{}^2/2(1 - \epsilon\zeta)] \tag{4}$$

$H(t)$ is the unit step function and $J_1(z)$ is the Bessel function of the first kind of order 1. It can be verified directly that Eq. (3) represents the solution for a point source in the medium defined by Eq. (1). Allowing ϵ to go to zero in Eq. (3) gives the following:

$$\lim_{\epsilon\to 0} p'(\xi,\eta,\zeta,t) = \frac{1}{4\pi\rho_1}\,\delta\left(t - \frac{\rho_1}{c_0}\right)$$

which is the familiar "fundamental solution" for the wave equation with constant sound speed c_0.

The term in Eq. (3) which involves the Bessel function may be interpreted as a "tail" effect because of the variation in the sound speed which is carried along with the "instantaneous" signal represented by the delta function. The existence of this tail of course verifies the fact that Huygens' Principle[9] is not valid for Eq. (2). The form of p' also shows that the wavefront of the source disturbance is given by $t - \nu/\epsilon c_0 = 0$. Using Eq. (4) this becomes

$$\xi^2 + \eta^2 + \left[\zeta + \frac{1}{\epsilon}(\cosh\epsilon c_0 t - 1)\right]^2 = \frac{1}{\epsilon^2}\sinh^2\epsilon c_0 t \quad (5)$$

Expression (5), for the wavefront due to a point disturbance in this medium, can be obtained without knowledge of the solution of (3) by using the theory of geometrical optics.[2,10] It therefore is seen that the disturbance is carried away from the origin on a spherical front whose center moves down the ζ axis as time increases; the top of the sphere asymptotically approaches the level $\zeta = 1/\epsilon$.

III. Extension to the Case of a Moving Source

Consider now a source starting instantaneously from rest at the origin and moving with a constant velocity V in the negative ξ direction. If the source strength K is independent of time, the pressure field is expressed by

$$p = K\int_0^t p'(\xi + V\tau, \eta, \zeta, t - \tau)d\tau \quad (6)$$

where $p'(\xi,\eta,\zeta,t)$ is given by Eq. (3). In order to study Eq. (6), it is convenient to define the following new variables: $x = \xi + Vt$, $y = \eta$, $z = \zeta$, and $\theta = t - \tau$. The variables x, y, z represent the coordinates of a point measured with respect to the moving source as origin, whereas θ is the time elapsed since emission of a particular disturbance at the instant τ. A unique value of θ corresponds to each elementary spherical front emitted during the motion of the source.

Equation (6) is written as $p = p_{\mathrm{I}} - p_{\mathrm{II}}$, and p_{I} may be simplified using the relation

$$\delta\left(\theta - \frac{\nu}{\epsilon c_0}\right) = \sum_{i=1}^{N}\delta(\theta - t_i)\left|1 - \frac{1}{\epsilon c_0}\frac{\partial\nu}{\partial\theta}\right|_{\theta=t_i}^{-1}$$

where the N values of t_i are the roots of $\theta - \nu/\epsilon c_0 = 0$. This gives

$$p_{\mathrm{I}} = \frac{\epsilon K(1-\epsilon z)^{1/2}}{4\pi}\sum_{i=1}^{N}|(1-\epsilon z)\sinh\epsilon c_0 t_i + \epsilon M_0(x - Vt_i)|^{-1} \quad (7a)$$

$$p_{\mathrm{II}} = \frac{K(1-\epsilon z)^{1/2}}{4\pi\epsilon}\int_0^t \frac{\nu J_1[(\epsilon c_0/2)(\theta^2 - \nu^2/\epsilon^2 c_0^2)^{1/2}]}{R_1 R_2(\theta^2 - \nu^2/\epsilon^2 c_0^2)^{1/2}} \times H(\theta - \nu/\epsilon c_0)d\theta \quad (7b)$$

where

$$R_1^2 = (x - V\theta)^2 + y^2 + z^2 \qquad R_2^2 + 4(1-\epsilon z)/\epsilon^2$$

$$\nu(\theta) = \cosh^{-1}[1 + \epsilon^2 R_1^2/2(1-\epsilon z)]$$

$M_0 = V/c_0$ is the Mach number at the source level $z = 0$ It will be shown later than N is constant at a point (x,y,z) for certain discrete intervals of time, so that the first term in Eq. (7) represents a quasi-steady effect, whereas the second forms an unsteady contribution for all time.

IV. Geometry of the Moving Disturbance

In terms of (x, y, z, θ), the individual spherical disturbances emitted at instants τ may be expressed as

$$\epsilon^2[(x - V\theta)^2 + y^2 + z^2] - 2(1-\epsilon z)(\cosh\epsilon c_0\theta - 1) = 0 \quad (8)$$

The equation of the envelope of this system of spherical fronts may be derived by eliminating the parameter θ between Eq. (8) and its derivative with respect to θ,

$$\epsilon M_0(x - V\theta) = -(1-\epsilon z)\sinh\epsilon c_0\theta \quad (9)$$

Further, the surface thus obtained possesses an edge of regression, and its equation is given by solving the derivative of Eq. (9) simultaneously with Eqs. (8) and (9). The edge of regression, i.e., a cuspidal ridge on the wave envelope, is given parametrically by

$$\begin{aligned}\epsilon x^* &= \epsilon V\theta - M_0\tanh\epsilon c_0\theta \\ \epsilon y^* &= \pm[(M_0^2 - 1)(1 - M_0^2\operatorname{sech}^2\epsilon c_0\theta)]^{1/2} \\ \epsilon z^* &= 1 - M_0^2\operatorname{sech}\epsilon c_0\theta\end{aligned} \quad (10)$$

In the plane $y = 0$, Eqs. (8) and (9) imply

$$\epsilon(x - V\theta) = \pm(\beta_0 \pm \beta) \quad (11)$$

where

$$\beta^2(z) = (1-\epsilon z)^2[M^2(z) - 1] = M_0^2 - (1-\epsilon z)^2$$

$$\beta_0^2 = \beta^2(0)$$

Equation (11), together with Eq. (9), yields the three branches $x = f_i(z)$ of the envelope on $y = 0$, as shown in Fig. 1, where

$$\begin{aligned}\epsilon f_1(z) &= -(\beta_0 - \beta) + M_0\sinh^{-1}[M_0(\beta_0 - \beta)/(1-\epsilon z)] \\ \epsilon f_2(z) &= -(\beta_0 + \beta) + M_0\sinh^{-1}[M_0(\beta_0 + \beta)/(1-\epsilon z)] \\ \epsilon f_3(z) &= +(\beta_0 - \beta) - M_0\sinh^{-1}[M_0(\beta_0 - \beta)/(1-\epsilon z)]\end{aligned} \quad (12)$$

Also, from Eq. (10), the cusp occurs on $y = 0$ at the point

$$\epsilon x_0^* = M_0\cosh^{-1}M_0 - \beta_0$$

$$\epsilon z_0^* = -(M_0 - 1)$$

and is formed at the instant $\theta = t^* = (1/\epsilon c_0)\cosh^{-1}M_0$.

Define the sonic level as the plane at which $M(z) = 1$; that is, the plane $z = -(M_0 - 1)/\epsilon$, $\beta(z) = 0$, at which the cuspidal ridge intersects the plane $y = 0$. Then Eq. (12) shows that there exists no envelope of the system of elementary spherical fronts emitted by the moving source unless $M_0 > 1$, and that this envelope lies entirely in the region $z > -(M_0 - 1)/\epsilon$. That is, the envelope exists only above the sonic level. Let the term "initial front" refer to that spherical front set off at $\tau = 0$ at the origin of the (ξ,η,ζ) system of coordinates. Then the value of θ corresponding to the initial front is $\theta = t$. For $0 < t < t^*$, the cusp has not formed in the plane $y = 0$, and the disturbed region is as shown in Fig. 1a. The signal is confined to the regions AOBCA inside the envelope and within the initial front. At $t = t^*$ the cusp forms at $z = -(M_0 - 1)/\epsilon$ and the disturbed region is as shown in Fig. 1b. For $t^* < t$ the third branch of the envelope develops away from the cusp along BD. The initial front shown, in part, as EDA in Fig. 1c propagates into the undisturbed region ahead of the cone-like envelope AOB. Segment BD of the envelope is formed by confluence of those elementary spherical fronts for which $t^* \leq \theta \leq t$. As t increases from t^*, the initial front advances up the envelope OB, overtaking the moving source at the point 0. In the limit as $t \to \infty$, the radius of the initial front becomes infinite, that front finally coinciding with the zero sound speed level $z = 1/\epsilon$ (Fig. 1d).

The cone-like surface shown as AOB in the plane, and the rear branch BD of the envelope, correspond to the Mach cone that is formed if $c(z)$ is constant. That is, the envelope is related to the characteristic surfaces of the wave equation written in the moving coordinate system:

$$[1 - M^2(z)]p_{xx} + p_{yy} + p_{zz} = \frac{1}{c^2(z)}P_{\theta\theta} + \frac{2V}{c^2(z)}p_{x\theta}$$

For the steady state, this equation is hyperbolic above the sonic level with characteristics defined by Eq. (12) in the plane $y = 0$. Below $z = -(M_0 - 1)/\epsilon$, however, the equation is elliptic. Hence, as is evident in Fig. 1, there can exist no steady-state solution to the present problem which involves an undisturbed region upstream of the Mach envelope, as in the case of constant sound speed.

As previously stated, N stands for the number of positive roots t_i of the equation $\theta - \nu(\theta)/\epsilon c_0 = 0$. In the plane $y = 0$, this is

$$\cosh \epsilon c_0 \theta = 1 + \{[\epsilon^2(x - V\theta)^2 + \epsilon^2 z^2]/[2(1 - \epsilon z)]\} \quad (13)$$

The number N for various regions of the flow in the plane and the boundaries of these regions may be determined by an analysis of the transcendental equation (13). It is found that N takes on the values 0,1,2, or 3, depending on the location $(x,0,z)$ and the value of t.

The roots t_i are, of course, the distinct values of θ which are attached to the N elementary waves, emitted at different instants τ, which just reach the point (x,y,z) at the instant t. Referring again to Fig. 1, it is possible to sketch the circles that form the wave pattern in the plane $y = 0$. N is then the number of these circles through a given point. Consider first Fig. 1a for $0 \leq t \leq t^*$. In this case, as the pattern builds up from the source point 0, it is readily seen that two circles pass through any point inside OABO, whereas only one passes through the points inside the initial front ABCA. N is zero for all other points, thus reflecting the fact that no signal has reached these points in the time t. For $t^* < t$ (Fig. 1c), the early signals $t^* < \theta$ sweep back across the region behind OB. It is then apparent that $N = 3$ inside EBDE; $N = 2$ inside OEDAO; and $n = 1$ for all other points interior to the circular initial front EDA. As $t \to \infty$ (Fig. 1d), $N = 3$ for all points interior to AOBD, whereas $N = 1$ in the entire remainder of the half-space.

The equations of the envelope may be interpreted as a definition of those points for which the right side of Eq. (13) is tangent to $\cosh \epsilon c_0 \theta$. That is, the envelope is the locus of points at which distinct roots t_i merge. It can be demonstrated that the cusp point $(x_0^*, 0, z_0^*)$ is the only point in the plane at which Eq. (13) has a single root $\theta = t^*$.

V. Discussion of Source Solution for Small ϵ

It is of value to determine the behavior of the pressure $p(x,y,z,t)$ as a function of the parameter ϵ when the parameter is small. For in any real atmosphere, the gradient of the sound speed is actually very small. The value ϵ corresponding to the standard atmosphere is of the order of 4×10^{-6} ft^{-1}. In addition, it is the objective of this analysis to determine the pressure field generated by a slender body of revolution moving at constant velocity and altitude in the medium defined by Eq. (1). The body will be represented by a line distribution of sources over the body length L. Then ϵ will be considered small if $\epsilon L \ll 1$; that is, if the distance from the body to the altitude at which $c(z) = 0$ is much greater than the length of the body.

Consider Eq. (13) for the roots t_i. It has been seen that for $0 < \theta < \infty$, $\epsilon \neq 0$, this equation possesses either three roots or one root depending on whether the point $(x,0,z)$ lies inside or outside the Mach surface shown in Fig. 1d. Also, for finite x and z, all three roots t_i are finite. In particular, at $x = z = 0$, Eq. (13) becomes

$$\cosh \epsilon c_0 \theta = 1 + \{[M_0^2(\epsilon c_0 \theta)^2]/2\} \quad (14)$$

Equation (14) has three roots for $0 \leq \theta$: a double root $\theta = 0$ corresponding to the source point, and a third root $\theta = t_3$ corresponding to the early signal that overtakes the source. Now define $X = x/L$, $Z = z/L$, $T = \epsilon c_0 \theta$, $T_i = \epsilon c_0 t_i$, and $\bar{\epsilon} = \epsilon L$. Then for $\bar{\epsilon} \ll 1$ with X and Z finite, Eq. (13) is

$$\cosh T - 1 - M_0^2 T^2/2 = 0(\bar{\epsilon}) \quad (15)$$

The order symbol $0(\bar{\epsilon})$ denotes a term $H(X,Z,T)$ such that $|H| < A\bar{\epsilon}$ where A is some positive number. Now for $\bar{\epsilon} \ll 1$, the roots T_i for nonzero X and Z must make the left side of Eq. (15) arbitrarily small, and the only way this can occur is for the roots of Eq. (15) to approach those of Eq. (14). In other words, for $\bar{\epsilon} \ll 1$ and inside the Mach envelope, the three roots of Eq. (13) are such that $T_{1,2} \to 0$, whereas T_3 remains finite. Outside the Mach surface, Eq. (13) has one finite root for all finite X and Z, so that in this region T_1 remains finite as $\bar{\epsilon} \to 0$.

The expression for the lower front portion of the Mach surface on $Y = 0$ from Eq. (12) is

$$\bar{\epsilon} X = -(\beta_0 - \beta) + M_0 \sinh^{-1}\{[M_0(\beta_0 - \beta)]/(1 - \epsilon Z)\}$$

Consider this expression for $\bar{\epsilon} \ll 1$ and Z finite. In this case, $\beta \cong \beta_0(1 + \bar{\epsilon} Z/\beta_0^2) + 0(\bar{\epsilon}^2)$, so that $X = -\beta_0 Z + 0(\bar{\epsilon})$.

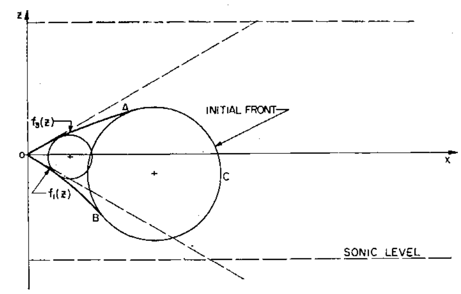

Fig. 1a Envelopes on $y = 0$ for $t < t^*$. Mach cone for uniform case shown in broken lines.

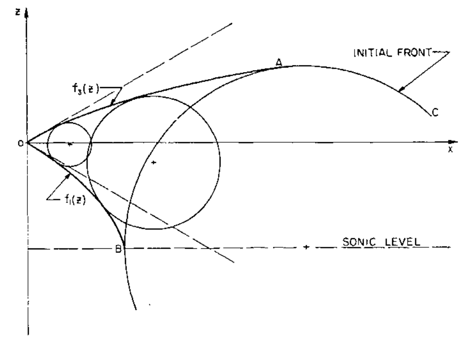

Fig. 1b Envelopes on $y = 0$ for $t = t^*$.

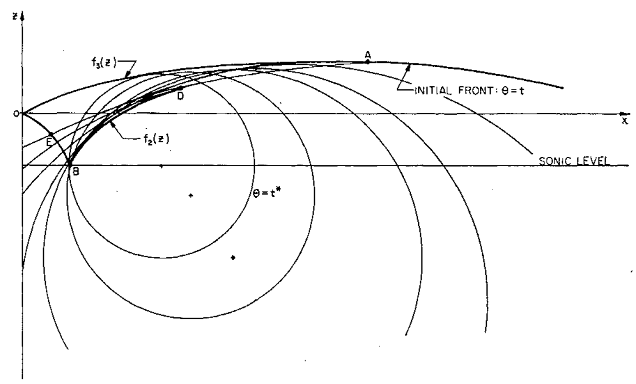

Fig. 1c Envelopes on $y = 0$ for $t^* < t$ showing circular fronts $t^* < \theta \leq t$ which propagate ahead of Mach surface.

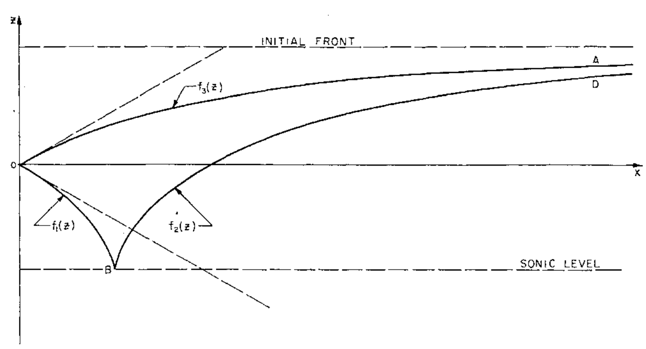

Fig. 1d Envelopes on $y = 0$ for $t \to \infty$. Mach cone for uniform case shown in broken lines.

Hence, as $\bar{\epsilon}$ becomes arbitrarily small, the Mach surface approaches arbitrarily closely the Mach cone $X^2 = \beta_0^2 Z^2$ of the constant case. At the same time, the cusp point (X_0^*, 0, Z_0^*) goes to infinity in X and Z. The third root t_3 within the envelope and the single root t_1 outside, both of which become arbitrarily large as $\bar{\epsilon} \to 0$, correspond to the early signal that propagates back across the Mach envelope after the cusp has formed. These roots become large for small $\bar{\epsilon}$ because the cusp forms after an increasingly long time.

The two roots $t_{1,2}$ of Eq. (13) which apply within the envelope can be determined approximately if $\bar{\epsilon} \ll 1$, since $T_{1,2}$ become small. For $T \ll 1$ and $\bar{\epsilon} \ll 1$, Eq. (13) is

$$1 + T^2/2 = 1 + [(\bar{\epsilon}X - M_0 T)^2 + (\bar{\epsilon}Z)^2]/2$$

to the lowest order in $\bar{\epsilon}$. Solving the quadratic equation yields

$$\beta_0^2 T_{1,2} = M_0 \bar{\epsilon} X \pm \bar{\epsilon}(X^2 - \beta_0^2 Z^2)^{1/2} \tag{16}$$

Now consider the pressure $p_{\rm I}$. Assume that t is large enough so that the region outside the Mach envelope is disturbed. Then from Eq. (7)

$$p_{\rm I} = \frac{\bar{\epsilon}K(1 - \bar{\epsilon}Z)^{1/2}}{4\pi L} \sum_{i=1}^{3} |(1 - \bar{\epsilon}Z)\sinh T_i + M_0(\bar{\epsilon}X - M_0 T_i)|^{-1} \tag{17}$$

in the interior of the Mach surface, where if $t < t_3$ it is to be understood that the third term in the summation is taken equal to zero. Similarly, in the exterior region if $t_1 < t$

$$p_{\rm I} = \frac{\bar{\epsilon}K(1 - \bar{\epsilon}Z)^{1/2}}{4\pi L} |(1 - \bar{\epsilon}Z)\sinh T_1 + M_0(\bar{\epsilon}X - M_0 T_1)|^{-1} \tag{18}$$

Now assume $\bar{\epsilon} \ll 1$. Since T_3 in Eq. (17) remains finite as $\bar{\epsilon}$ becomes small, while $T_{1,2}$ become small, the right side of (17) is

$$\frac{\bar{\epsilon}K(1 - \bar{\epsilon}Z)^{1/2}}{4\pi L} \{1(1 - \bar{\epsilon}Z)T_1 + M_0(\bar{\epsilon}X - M_0T_1)|^{-1} + |(1 - \bar{\epsilon}Z)T_2 + M_0(\bar{\epsilon}X - M_0T_2)|^{-1}\} + 0(\bar{\epsilon})$$

Using Eq. (16) for T_1 and T_2 yields

$$p_{\rm I} = (K/2\pi)(x^2 - \beta_0^2 z^2)^{-1/2} + 0(\bar{\epsilon}) \tag{19}$$

For $\bar{\epsilon} = 0$, this result is exactly the pressure for steady motion in the case of constant sound speed c_0. Similarly, for the exterior pressure [Eq. (18)], since T_1 remains finite as $\bar{\epsilon}$ becomes small, $p_{\rm I}$ in this region is $0(\bar{\epsilon})$ and vanishes completely for $\bar{\epsilon} = 0$.

The remaining contribution to the pressure field is made by $p_{\rm II}$, which corresponds to the tail effect given by the integral in Eq. (7). It can be shown[8] that this integral is proportional to $\bar{\epsilon}$ for all finite X, Y, Z, T. Hence, the entire contribution of $p_{\rm II}$ to the pressure field for $\bar{\epsilon} \ll 1$ is $0(\bar{\epsilon})$ compared to that of $p_{\rm I}$ and is negligible for sufficiently small $\bar{\epsilon}$.

VI. Singularity of the Source Solution

At every point of the Mach surface [AOBD, Fig. 1] there is a double root of Eq. (8) which corresponds to the two elementary fronts that merge to form the envelope. This value of θ satisfies Eq. (9) as well as Eq. (8). Therefore, it is seen from Eq. (7) that the instantaneous pressure $p_{\rm I}$ becomes infinite at each point of the envelope. This singularity of course conforms to the expected behavior of the fundamental solution for a hyperbolic partial differential equation. In view of the occurrence of the singularity over the entire Mach surface, it becomes essential to examine the behavior of the pressure $p_{\rm I}$ near the envelope, particularly in the vicinity of the cusp.

Consider a point P located at $(f_1 + \delta_1, 0, z)$. For $\theta < t^*$ there exist two roots t_i at P which merge to a single value t_{f_1} as δ_1 approaches zero. From Eqs. (11) and (12) the time t_{f_1} corresponding to the point $(f_1,0,z)$ on OB is

$$t_{f_1} = \frac{1}{\epsilon V}(\epsilon f_1 + \beta_0 - \beta) = \frac{1}{\epsilon c_0}\sinh^{-1}\frac{M_0(\beta_0 - \beta)}{1 - \epsilon z}$$

For $t^* < \theta$ and θ large enough so that early signals (Fig. 1c) have swept across P, there exist three roots t_i: the two described previously and a third that does not satisfy Eq. (16) on OB and, hence, does not contribute to the singularity there. Let $\alpha = \epsilon c_0(t\epsilon_1 - \theta)$. Then Eq. (13) can be written

$$(M_0^2 - \beta_0\beta)(\cosh\alpha - 1) - M_0(\beta_0 - \beta)(\sinh\alpha - \alpha) - \{[(\epsilon\delta_1 + M_0\alpha)^2]/2\} + \epsilon\delta_1(\beta_0 - \beta) = 0 \tag{20}$$

Solutions of Eq. (20) are $\alpha_i = \epsilon c_0(t_{f1} - t_i)$. Let $\lambda_1 = (\epsilon\delta_1)^{1/2}$. Then Eq. (20) can be written

$$F(\alpha,\lambda_1) = 0 \tag{21}$$

which defines a function $\alpha(\lambda_1)$ such that, if $\beta \neq 0$, $\lambda_1 = 0$ is a double root of $\alpha(\lambda_1)$. Differentiating Eq. (21) twice with respect to λ_1 it is found that at $\lambda_1 = 0$,

$$(d\alpha/d\lambda_1)^2 = [2(\beta_0 - \beta)]/\beta_0\beta$$

and hence for $\delta_1 \ll 1/\epsilon$, $0 < \beta$,

$$\alpha_{1,2}(\delta_1) \cong [2(\beta_0 - \beta)\epsilon\delta_1/\beta_0\beta]^{1/2} \tag{22}$$

Let $p_{{\rm I}_s}$ denote the singular part of the pressure $p_{\rm I}$. In terms of the α_i it may be written as

$$p_{{\rm I}_s} = \frac{\epsilon K(1 - \epsilon z)^{1/2}}{4\pi} \sum_{i=1}^{2} |M_0(\beta_0 - \beta)\cosh\alpha_i - (M_0^2 - \beta_0\beta)\sinh\alpha_i + M_0(\epsilon\delta_1 + M_0\alpha_i + \beta - \beta_0)|^{-1}$$

Then for $\epsilon\delta_1 \ll 1$ using Eq. (22),

$$p_{{\rm I}_s} = \{[\epsilon K(1 - \epsilon z)^{1/2}]/2\pi\}[2\beta_0\beta(\beta_0 - \beta)\epsilon\delta_1]^{-1/2} \tag{23}$$

To determine the form of $p_{\rm I}$ near the cusp point where $\beta = 0$, it is necessary to note that δ_1 must vanish with β if the point P is to remain within the region AOBD. Let d be defined such that $d = f_2(z) - f_1(z)$. From Eq. (12) $\epsilon d = 2M_0 \tanh^{-1}\beta/M_0 - 2\beta \approx 2\beta^3/3M_0^2$. Now let the field point P be located such that $\delta_1 = hd$ where $0 \leq h \leq 1$. Then Eq. (20) may be written as $F(\alpha,\beta) = 0$, which defines a function $\alpha(\beta)$ for which $\alpha(0) = 0$ and which possesses three distinct tangents given by

$$[(d\alpha/d\beta)^3] + (3/M_0)[(d\alpha/d\beta)^2] - (4h/M_0^3) = 0 \tag{24}$$

The three real roots of Eq. (24) are $\alpha_i'(0) = [\sigma_i(h) - 1]/M_0$, $i = 1,2,3$, where

$$\sigma_i(h) = 2\cos\left[\frac{\gamma}{3} + (i - 1)\frac{2\pi}{3}\right] \tag{25}$$

and

$$\cos\gamma = 2h - 1 \tag{26}$$

Thus for $\beta \to 0$, the $\alpha_i(\beta)$ are, approximately,

$$\alpha_i(\beta) \cong (\beta/M_0)[\sigma_i(h) - 1] \qquad i = 1,2,3 \tag{27}$$

For $\beta \neq 0$, two of the roots α_i are zero for $h = 0$ or $h = 1$, corresponding to the double roots already known to occur on the envelope away from the cusp.

The pressure $p_{\rm I}$ can be written in terms of α_i as before, and for $\beta \ll 1$ it becomes

$$p_{\rm I} \cong \frac{\epsilon K M_0^{3/2}}{2\pi\beta_0\beta^2} \sum_{i=1}^{3} |\sigma_i^2(h) - 1|^{-1} \tag{28}$$

To determine corresponding expressions valid outside the envelope near the cusp point $(x_0^*, 0, z_0^*)$, let $\delta_2 = f_1(z) - x$. It is found that Eq. (13) written as $F(\alpha,\beta,\lambda_2) = 0$ where $\lambda_2 = (\epsilon\delta_2)^{1/3}$ then defines a function $\alpha(\beta,\lambda_2)$ for which $\alpha(0,0) = 0$ and for which a unique tangent plane exists at (0,0). Hence, for $\beta \ll 1$ and $\delta_2 \ll 1/\epsilon$, α may be written

$$\alpha \cong -[(6\epsilon\delta_2/M_0)^{1/3}] - (3\beta/M_0)$$

For $\delta_2 = 0$, this single root α agrees with the nonzero root given on the envelope ($h = 0$) by Eq. (27). Then for $\beta \ll 1$ and $\delta_2 \ll 1/\epsilon$,

$$p_{\mathrm{I}} \cong [(\epsilon K M_0^{1/6})/2\pi\beta_0]|(6\epsilon\delta_2)^{2/3} + 4(6\epsilon\delta_2)^{1/3}\beta/M_0^{2/3} + 3\beta^2/M_0^{4/3}|^{-1} \quad (29)$$

For $\delta_2 = 0$ and $\beta \neq 0$ this expression is finite and equal to the single term $i = 2$ in Eq. (28), which shows that the nonsingular part of p_{I} is continuous across the envelope OB. In exactly the same manner, it is found that for $\beta \ll 1$ and $\delta_3 = x - f_2(z) \ll 1/\epsilon$,

$$p_{\mathrm{I}} \cong [(\epsilon K M_0^{1/6})/2\pi\beta_0]|(6\epsilon\delta_3)^{2/3} + 4(6\epsilon\delta_3)^{1/3}\beta/M_0^{2/3} + 3\beta^2/M_0^{4/3}|^{-1} \quad (30)$$

VII. Flow Past a Slender Body

The steady pressure field $p_c(x,0,z)$ corresponding to Eq. (7), which is generated by a supersonic source in a medium in which the sound speed is a constant c_0, is given by Eq. (19) with $\bar{\epsilon} = 0$.[11] Let a slender pointed body of revolution be situated on $0 \le x \le L$ with the body surface given by $r = R_0(x)$ for $0 \le x \le L$ with $R_0(0) = 0$. The body may be represented by an axial distribution of supersonic sources whose strength per unit length is $\rho_0 V^2 S''(x)$, where $S(x) = \pi R_0^2(x)$ is the distribution of cross-sectional area of the slender body. Let $x = \delta + \beta_0 z$ and assume $L/\beta_0 z \ll 1$. Then

$$P_c(x,0,z) \cong \frac{\rho_0 V^2}{2\pi(2\beta_0 z)^{1/2}} \int_0^\delta \frac{S''(x')dx'}{(\delta - x')^{1/2}} \qquad \text{for } 0 < \delta \quad (31)$$

Consider a line distribution of sources defined by Eq. (7) along $0 \le x \le L$ where $\bar{\epsilon} \ll 1$. In view of Eq. (19), for sufficiently small $\bar{\epsilon}$, the slender body will be represented in the nonuniform atmosphere by a source strength $\rho_0 V^2 S''(x)$ within an error consistent with the linearized theory. Hence, for the plane $y = 0$ in the nonuniform atmosphere

$$P(x,0,z,t) = \rho_0 V^2 \int S''(x')\bar{p}(x - x',0,z,t)dx'$$

where ρ_0 is the rest density at $z = 0$ and $K\bar{p}$ is defined in Eq. (7).

As in Sec. III, let P be written as $P_{\mathrm{I}} - P_{\mathrm{II}}$ where P_{I} is that portion of P derived from the quasi-steady part of p. That is

$$P_{\mathrm{I}}(x,0,z,t) = \frac{\rho_0 V^2 \epsilon(1 - \epsilon z)^{1/2}}{4\pi} \times \int \frac{S''(x')\,dx'}{|(1 - \epsilon z)\sinh \epsilon c_0 t_i + \epsilon M_0(x - x' - Vt_i)|} \quad (32)$$

where the t_i now are functions of $x - x'$.

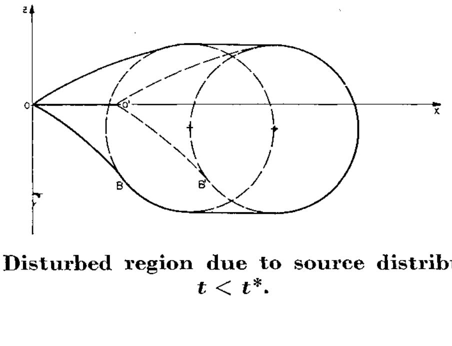

Fig. 2 Disturbed region due to source distribution for $t < t^*$.

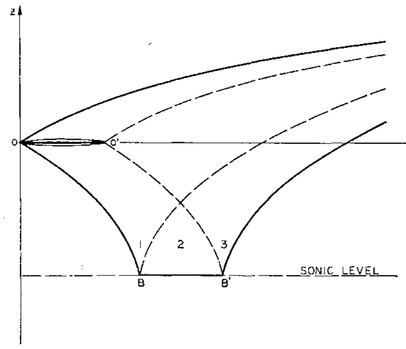

Fig. 3 Disturbed region due to source distribution for $t^* < t$.

For $0 < t < t^*$ the disturbance from the source distribution is confined, as shown in Fig. 2. Consider a point $(x,0,z)$ for which $-(M_0 - 1)/\epsilon < z < 0$ and $x - f_1(z) \ll 1/\epsilon$. The integrand in Eq. (32) is zero for $x - x' < f_1(z)$, so that the domain of dependence of the point x' on the initial line is $0 \le x' \le x - f_1(z)$. Then using Eq. (23), to the first approximation

$$P_{\mathrm{I}}(x,0,z,t) = \frac{\rho_0 V^2}{2\pi}\left[\frac{\epsilon(1 - \epsilon z)}{2\beta_0\beta(\beta_0 - \beta)}\right]^{1/2} \int_0^{x-f_1} \frac{S''(x')dx'}{(x - x' - f_1)^{1/2}}$$

for (33)

$$0 \le x - f_1 \le L$$

For $t^* \ll t$, P_{I} at the point $(x,0,z)$ described previously is given by Eq. (33) plus another term corresponding to $i = 3$ in Eq. (32). This term is a continuous function across the envelope OB which is simply superposed on that given by Eq. (33). For a small sound speed gradient, Eq. (33) reduces to Eq. (31), showing that for points far from the body but well above the cusp level, the pressure field in the variable atmosphere is not significantly different from that in the constant case.

For the purpose of studying P_{I} near the cusp level $z = -(M_0 - 1)/\epsilon$, assume $t^* < t$. Then the envelopes will be developed, as shown in Fig. 3, with the disturbance extended into the region directly ahead of OB. The locus of cusps BB′, represents a caustic of the ray system emitted by the travelling source distribution and it is necessary to determine whether Eq. (32) is integrable as $(x,0,z)$ approaches this locus. Let the field point be located at a level z for which $\beta \ll 1$. For a single source at $x = x'$, the pressure p_{I} is described by Eq. (28) with x replaced by $x - x'$ whenever $f_1(z) \le x - x' \le f_2(z)$, whereas one of Eqs. (29) and (30) holds outside the interval. That is, the pressure p_{I} due to a single source may be considered as being composed of a part p_{I_1}, which is confined inside the Mach envelopes and a part p_{I_2} that exists outside those surfaces.

Let P_{I} be written as $P_{\mathrm{I}} = P_{\mathrm{I}_1} + P_{\mathrm{I}_2}$ where P_{I_1} represents the distributed effect of p_{I_1}, and P_{I_2} stands for that due to p_{I_2}. By definition, $p_{\mathrm{I}_1}(x,0,z,t)$ vanishes outside the interval $f_1(z) \le x \le f_2(z)$ so that for $t^* < t$ using Eq. (28)

$$P_{\mathrm{I}_1}(x,0,z,t) = \frac{\rho_0 V^2 \epsilon M_0^{3/2}}{2\pi\beta_0\beta^2} \int_{l_1}^{l_2} \sum_{i=1}^{3} \frac{S''(x')dx'}{|\sigma_i^2(x - x' - f_1) - 1|} \quad (34)$$

where

$l_1 = 0,\ l_2 = x - f_1(z)$ for $f_1(z) \le x \le f_2(z)$

$l_1 = x - f_2(z),\ l_2 = x - f_1(z)$ for $f_2(z) \le x \le f_1(z) \times L$

$l_1 = x - f_2(z),\ l_2 = L$ for $f_1(z) + L \le x \le f_2(z) + L$

The three regions in Eq. (34) are shown as 1, 2, and 3 in Fig.

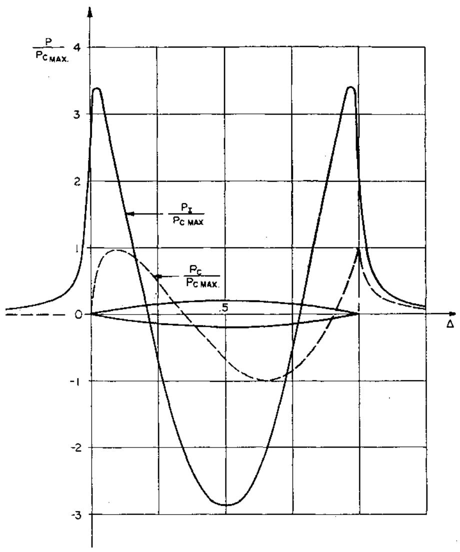

Fig. 4 Pressure signature at cusp level for parabolic body with $M_0 = 1.17$.

3. By integrating by parts it can be established[8] that P_I, is proportional to β. Therefore, for $z = z_0^*$ the pressure $P_{I_1}(x, 0, z_0^*, t) = 0$ along the cusp locus $f_1(z_0^*) \leq x \leq f_1(z_0^*) + L$. On the other hand, the behavior of P_I for $z_0^* < z$ is given by Eq. (33), and is zero on the front $x = f_1(z)$. This expression can also be considered as generated by those portions of the individual source signals that are confined inside their own Mach surfaces, and thus reach the envelope OB directly without propagating through the region below the sonic line. Hence, the zero in the pressure over the line BB′ in Fig. 3 is simply an extension of this "direct" pressure field over the entire front OB.

The pressure P_{I_2} is the total effect of $p_{I_2}(x - x', 0, z, t)$. For $-(M_0 - 1)/\epsilon < z$, these pressure signals are nonzero only outside the interval $f_1(z) \leq x - x' \leq f_2(z)$. It is of primary interest in this analysis to determine the behavior of $P_I = P_{I2}$ along the sonic line. Therefore, consider the expression for P_{I_2} at the level $z = -(M_0 - 1)/\epsilon$, $\beta = 0$. From Eqs. (29) and (30), if $t^* < t$ so that the cusp has formed,

$$P_I(x,0,z_0^*,t) = \frac{\rho_0 V^2 \epsilon^{1/3} M_0^{1/6}}{2\pi(6)^{2/3}\beta_0} \int_0^L \frac{s''(x')dx'}{|x - x_0^* - x'|^{2/3}} \quad (35)$$

Equation (35) represents the major result of this analysis. At the cusp level directly beneath the moving slender body, the pressure field consists of the expression (35) plus a time-dependent correction of $0(\bar{\epsilon})$. This correction is due to the tail effect of those signals $t^* \leq 0 \leq t$ that have propagated upstream around the cusp associated with the moving source from which they were emitted. If Eq. (35) is nondimensionalized with respect to L, as in Sec. V, then, to the lowest order in $\bar{\epsilon}$, $P(X,0,Z_0^*,t)$ is proportional to $(\bar{\epsilon})^{1/3}$. At the corresponding distance from the body in a constant atmosphere, the pressure P_c is proportional to $(\bar{\epsilon})^{1/2}$ (see Sec. VIII). Hence, it is seen that the refractive property inherent in the linearly varying atmosphere, which gives rise to a focussing of pressure waves at the sonic level, results in a pressure signature at that level whose magnitude is greater than that of the constant case by a factor proportional to $(\bar{\epsilon})^{-1/6}$. The examples given in the following section will exhibit clearly this effect of atmospheric refraction.

VIII. Examples of Pressure Signature at the Cusp Level

A. Body of Parabolic Profile

Let a slender body be defined by $R_0(x) = (2t_0x/L)(L - x)$ for $0 \leq x \leq L$ with the thickness ratio $t_0 \ll 1$. Then in di-

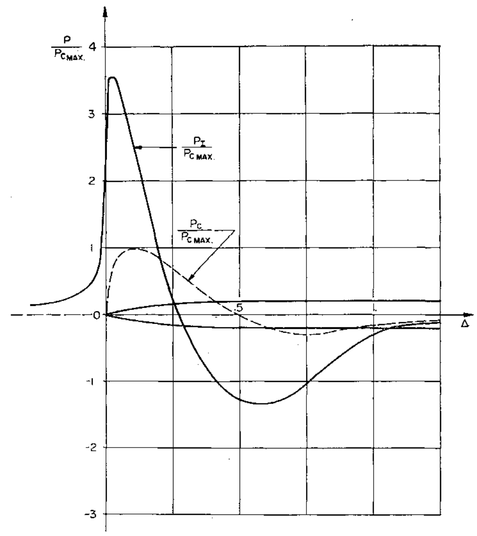

Fig. 5 Pressure signature at cusp level for body-wake combination with $M_0 = 1.17$.

mensionless form $S''(X) = 8\gamma t_0^2(1 - 6X + 6X^2)$. If $X = \beta_0 Z_0^* + \Delta$, then Eq. (31) yields

$$P_c(\Delta,0,Z_0^*) = (4\rho_0 V^2 t_0^2/5)\{[2\bar{\epsilon}/\beta_0(M_0 - 1)]^{1/2}\} \times G_{c_1}(\Delta) = P_{c_{\max}} G_{c_1}(\Delta) \quad (36)$$

where

$$G_{c_1}(\Delta) = \begin{cases} 0 & \text{for } \Delta < 0 \\ \Delta^{1/2}(5 - 20\Delta + 16\Delta^2) & \text{for } 0 \leq \Delta \leq 1 \\ \Delta^{1/2}(5 - 20\Delta + 16\Delta^2) - (\Delta - 1)^{1/2} \times [5 + 20(\Delta - 1) + 16(\Delta - 1)^2] & \text{for } 1 < \Delta \end{cases}$$

For the nonuniform case, Eq. (35) gives for $t^* < t$

$$P(X,0,Z_0^*) = \frac{2\rho_0 V^2 t_0^2}{\beta_0} (6\bar{\epsilon})^{1/3} M_0^{1/6} G_{v_1}(\Delta)$$

where

$$G_{v1}(\Delta) = \Delta^{1/3}(1 - 9\Delta/2 + 27\Delta^2/7) + (1 - \Delta)^{1/3}(5/14 - 45\Delta/14 + 27\Delta^2/7)$$

If Γ is defined as

$$\Gamma = [M_0^{1/3}(M_0 - 1)/2\beta_0]^{1/2}(\bar{\epsilon})^{-1/6}$$

the cusp level pressure signature can be expressed as

$$P(X,0,Z_0^*) = 5(3/4)^{1/3} P_{c_{\max}} \Gamma G_{v_1}(\Delta) \quad (37)$$

B. Body with Uniform Wake

Consider a body-wake combination defined by

$$R_0(x) = \begin{cases} (t_0/2L^2)[L^3 - (L - x)^3] & \text{for } 0 \leq x \leq L \\ t_0 L/2 & \text{for } L < x \end{cases}$$

Using the notation of the previous example,

$$P_c(\Delta,0,Z_0^*) = [(3\rho_0 V^2 t_0^2)/4]\{[2\bar{\epsilon}/\beta_0(M_0 - 1)]^{1/2}\} \times G_{c2}(\Delta) = 1.621 P_{c_{\max}} G_{c_2}(\Delta) \quad (38)$$

where

$$G_{c_2}(\Delta) = \begin{cases} 0 & \text{for } \Delta < 0 \\ \Delta^{1/2}(3 - 12\Delta + 16\Delta^2 - 64\Delta^3/7 + 128\Delta^4/63) & \text{for } 0 \leq \Delta \leq 1 \\ \Delta^{1/2}(3 - 12\Delta + 16\Delta^2 - 64\Delta^3/7 + 128\Delta^4/63) - (\Delta - 1)^{1/2}[4(\Delta - 1)/3 + 128(\Delta - 1)^4/63] & \text{for } 1 < \Delta \end{cases}$$

From Eq. (35) for the nonuniform case,

$$P(\Delta,0,Z_0^*) = 1.621(3/4)^{1/3}P_{c_{\max}}\Gamma G_{v_2}(\Delta) \qquad (39)$$

where

$$G_{v_2}(\Delta) = \Delta^{1/3}(3 - 27\Delta/2 + 135\Delta^2/7 - 81\Delta^3/7 + 243\Delta^4/91) - (1 - \Delta)^{1/3}(213/182 - 1671\Delta/182 + 1458\Delta^2/91 - 972\Delta^3/91 + 243\Delta^4/91)$$

The pressure signatures (36, 37, 38, and 39) are shown in Figs. 4 and 5 for a particular choice of parameters to be discussed in the following section.

IX. Results and Discussion

The results of the preceding section illustrate the amplification of the pressure signature which occurs in the nonhomogeneous atmosphere. They are not significantly altered if the body shape is taken as some other smooth function. Let the cusp occur at an altitude at which $c(z)$ has some fixed value: for example, at the ground level beneath the moving body. If c_g denotes this value, then $c_g = c_0 - az_0^*$, and the parameter $\epsilon = a/c_0$ may be expressed as $\epsilon = aM_0/c_g$. Then Γ becomes

$$\Gamma = (c_g/8aL)^{1/6}[(M_0 - 1)/(M_0 + 1)]^{1/4}$$

The speed of sound at the ground will be assumed to be 1117 fps with $a = 0.004$ sec^{-1} to correspond to the standard atmosphere.[3] The pressure signatures (37) and (39), as well as the corresponding expressions (36) and (38) for the constant case, are shown in Figs. 4 and 5 for $M_0 = 1.17$. This value of M_0 is such that $z_0^* = -40{,}000$ ft. The body profiles are drawn on the same figures in proper proportion, the scale being that of Δ, with the thickness ratio chosen as 0.1 for the purpose of the figures. Thus, as shown in Figs. 4 and 5, the maximum pressure rise behind the Mach surface at the cusp level in the nonhomogeneous atmosphere is greater than the maximum rise behind the Mach cone at the same level in the constant atmosphere by factors of 3.44 for the parabolic shape and 3.56 for the body-wake combination. The symmetry apparent in Fig. 4 for the parabolic body results from the fact that the elementary compression signals that pile up at the cusp are of the nature of subsonic disturbances, having propagated through the region outside the Mach surface. Hence a point on the sonic level has as its domain of dependence the entire length of the body, and a symmetric pressure signature exists if the body itself is symmetric.

The present state of experimental results is relatively incomplete in that virtually no measurements of pressure focussing exist which can be definitely attributed to atmospheric refraction. However, isolated instances of cusp pressure signatures (superbooms) have been recorded.[4,5] These measurements indicate a pressure buildup at the cusp of from 2 to 4 times; this factor is relative to pressures that occur far from the cusp where the classical theory gives a good estimate of the magnitude of the pressure rise. This significant amplification is not predicted by the existing techniques that are based for the most part upon geometrical acoustics concepts. Further experiments would be of interest in verifying the validity of the hypothesis that the results of a linear theory predict the behavior of the pressure field in a nonhomogeneous medium near the cusp.

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